§4. Topological Quantum Computation with Surface Codes

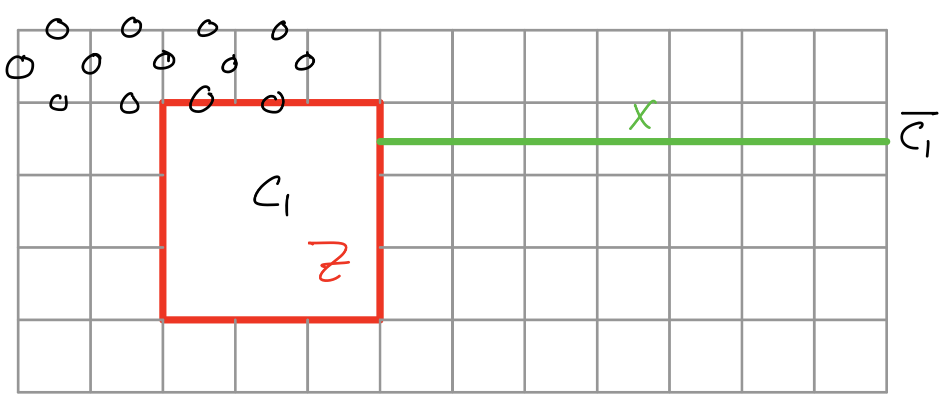
In this chapter we focus on the toric code model.

§4.1 Defect Pair Qubits

In order to realize N qubits, we need a surface of higher genus

→ use a large planar surface and punch holes on it "defects"

→ logical operators are realized by cycle surrounding defect and chain connecting defect to boundary



Define two defect regions $D \in C_2$ and $D' \in C_2$

→ remove all Z-type stabilizers inside these regions, i.e. $\{A_m\}_{f_m \in D \cup D'}$

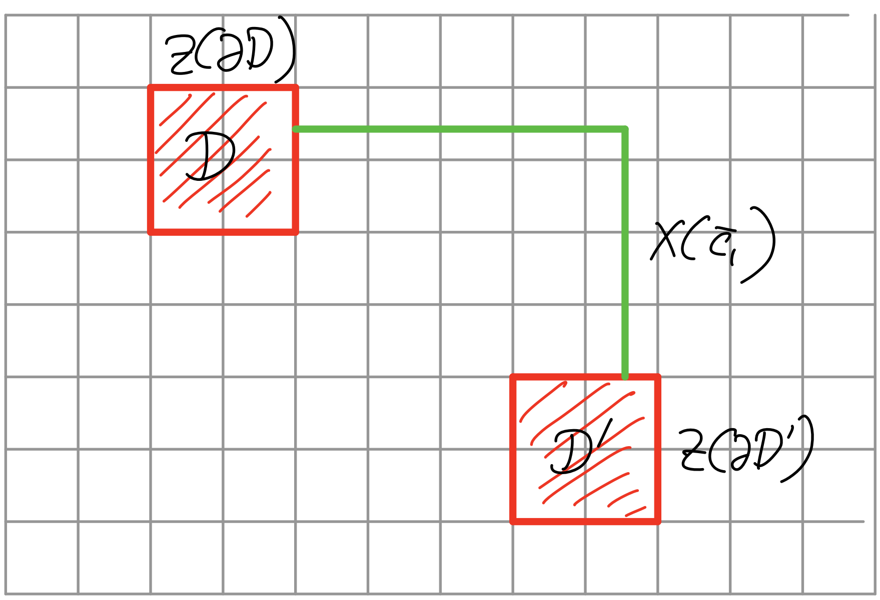
→ $Z(\partial D)$ and $Z(\partial D')$ are not stabilizer operators

→ add $Z(\partial D + \partial D')$ as stabilizer generator

→ choose $Z(\partial D)$ as a logical operator

as $Z(\partial D + \partial D')$ is stabilizer, we have

$$Z(\partial D) \sim Z(\partial D')$$

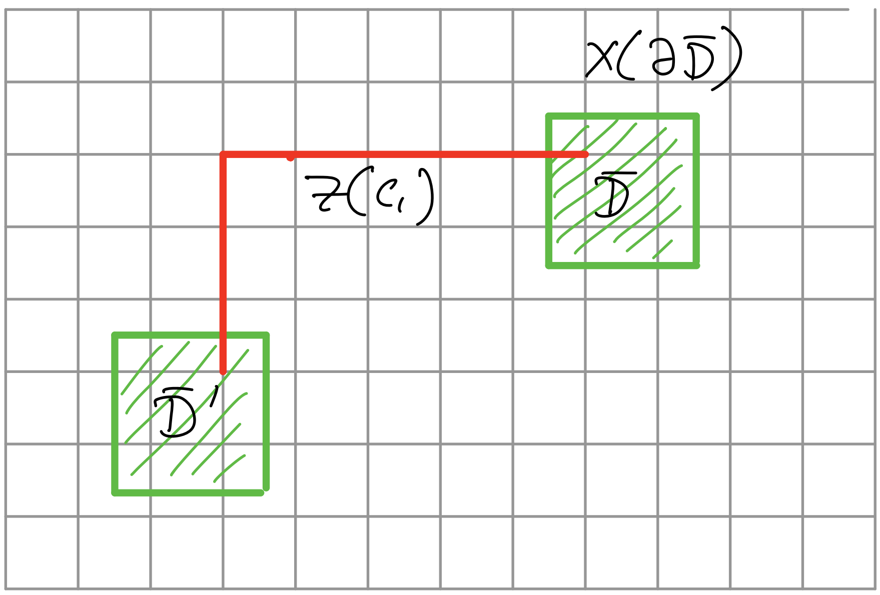


"primal defect pair qubit"

→ logical X operator is $X(\bar{c}_1)$, with $\bar{c}_1$

connecting the two defects

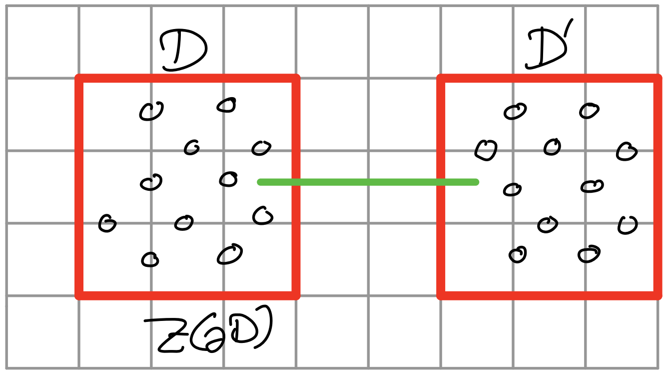
Similarly, we can also define a logical qubit by removing the star (X-type) stabilizers on dual defect regions $\bar{D}$ and $\bar{D}'$:



"dual defect pair qubit"

§4.2 Defect Creation, Annihilation, and Movement

Defect creation is accomplished by measuring the qubits inside the defect region D (not the boundary), in X-basis



→ measurment removes the plaquette operators inside D from stabilizer group ($[X, Z] \neq 0$)

→ post-measurement state is given by $|+\rangle$ for all measured qubits, stabilized by $Z(D)$

→ do the same for region D'

→ resulting state stabilized by $Z(\partial D + \partial D')$

Defect annihilation:

annihilation is executed by measuring plaquette operators inside region D

→ restores to stabilizer group

namely, before measurement we have

$$|D\rangle \sim \prod_{e \in D} \left(\frac{I + X_e}{2} \right) |v\rangle$$

where $|v\rangle$ indicates surface code without defect, i.e. vacuum

→ $|D\rangle$ is superposition of all possible X-op. applications

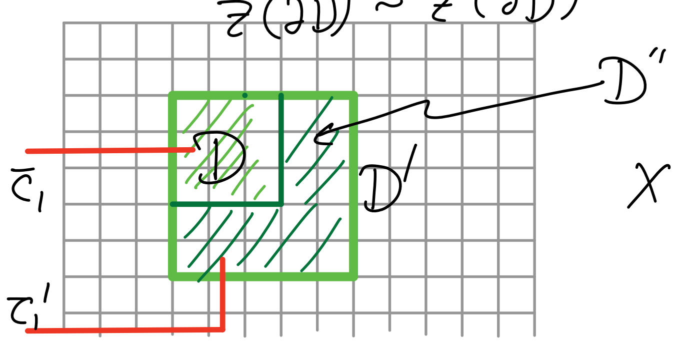
→ measurement of plaquette operators collapses superposition

→ parity of all measurement outcomes of plaquette operators inside D corresponds to eigenvalue of $Z(\partial D)$:

$$Z(\partial D) = \prod_{f_m \in D} Z(\partial f_m) = \prod_{f_m \in D} A_m$$

Suppose defect region D inside D' is annihilated by measuring qubits inside D

$Z(\partial D) \sim Z(\partial D')$



$X(\bar{c}_1) \sim X(\bar{c}_1')$

→ obtain eigenvalue $(-1)^m$ of $Z(\partial D)$

Let D'' be complement of D in D'

as $$Z(\partial D') = Z(\partial D'')\, Z(\partial D),$$

depending on the eigenvalue $(-1)^m$, the operator $(-1)^m Z(\partial D')$ acts as logical operator of defect D''

→ D' is contracted into D'' (without changing stored quantum inf.)

Defect movement on surface is implemented by combining expansion and contraction: